

A Geometrical Error in Some Computer Programs based on the Aki-Christofferson-Husebye (ACH) Method of Teleseismic Tomography

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Abstract Some computer programs based on the Aki-Christofferson-Husebye (ACH) method of teleseismic tomography contain an error caused by identifying local grid directions with azimuths on the spherical Earth. This error, which is most severe in high latitudes, introduces systematic errors into computed ray paths and distorts inferred Earth models. It is best dealt with by explicitly correcting for the difference between true and grid directions. Methods for computing these directions are presented in this article and are likely to be useful in many other kinds of regional geophysical studies that use Cartesian coordinates and flat-earth approximations.

The ACH Method

The ACH method, developed by Aki, Christofferson, and Husebye (Aki *et al.*, 1977) was the first application of tomography in seismology and has by now been used to study the three-dimensional structure of the crust and upper mantle beneath seismic arrays in dozens of places (Evans and Achauer, 1993). The method takes as input a collection of arrival-time residuals at the sensors of the array from a suite of distant earthquakes in various directions and generates estimates of the wave speeds in a number of rectangular blocks under the array.

We recently discovered, in a study of the structure beneath Iceland (Foulger *et al.*, 2000), that some computerized versions of the ACH method, including the program THRD (Evans and Achauer, 1993), contain an error in their treatment of azimuths, which increases with the size of the region under study and is particularly severe at high latitudes. This error leads to incorrect computed ray paths, which in turn distort the derived three-dimensional structure. This note explains the origin and consequences of this error and presents ways of correcting it. The mathematical methods presented are likely to be useful in other geophysical problems that approximate a region of the Earth as being flat.

The Problem

Local Grid

The ACH method represents the structure in a region using a local rectangular coordinate system with its origin in the center of the region being investigated. Establishing such a local coordinate system requires mapping points from the Earth's surface onto a plane, and there are many different ways in which this can be done. We will restrict our consideration to azimuthal mappings (those that preserve directions from the coordinate origin to all mapped points), but there

are still several different mappings in common use, which differ in the way they transform distances from the origin (Richardus and Adler, 1972; Snyder, 1982). Azimuthal mappings likely to be useful in seismological and other geophysical studies include the equidistant projection, which preserves distance from the origin, the equal-area projection, which preserves area, the stereographic projection, which is conformal (preserves shapes locally), the orthographic projection, in which points are projected along lines parallel to the vertical through the origin (the z axis), and the gnomonic projection, in which great circles become straight lines. Aki and his coworkers apparently used a mapping, based on approximate formulas for distance and azimuth due to Richter (1958, Appendix XII) and designed for hand computation. This mapping is not truly azimuthal and is defined in a way that makes it difficult to analyze mathematically, but it is approximately the same as the azimuthal-equidistant mapping.

Grid Directions Versus True Directions

In its original form, the ACH method computes ray paths using the same station-to-epicenter azimuth, in local coordinates, for all stations observing a particular earthquake. This practice, which is equivalent to approximating the wavefront on the ground as a straight line in the local coordinate system, has been followed in some other ACH computer programs, such as THREED (Ellsworth and Koyanagi, 1977; Zandt, 1981). At least one widely used program, however, THRD, has "improved" this approximation by using the true spherical azimuth at each station. Because of convergence of the meridians, this practice introduces into each azimuth an error equal to the difference between the local grid direction and the true azimuth, which at high latitudes can greatly exceed the errors related to wavefront curvature that motivated the modification.

Figure 1 illustrates the nature of the error. The black arrows represent the directions of propagation of teleseismic body waves arriving from earthquakes to the north and south. (For an epicentral distance of 90° , the intersection of the wavefront with the Earth's surface is a great circle, and this is the case illustrated.) Although the wavefronts are straight (great circles) in Figure 1, the azimuths of various points on them differ, because of convergence of the meridians on the spherical Earth. These are the station-to-epicenter azimuths that are given by standard computer programs, but if they are identified with the azimuths in the local grid, severe distortion of the wavefronts results. Wavefronts from the north converge artificially, while those from the south diverge. (This is in the northern hemisphere; the situation is reversed in the south.) Wavefronts arriving from the east and west are distorted even more severely.

This error causes the modified ACH algorithm to trace rays incorrectly, and to misunderstand the effects upon travel times of changing the wave speeds in the various blocks in the model, and results in distortion of derived models.

The Solutions

Plane-Wave Approximation

The simplest way to reduce the distortion is to revert to the original practice of Aki *et al.* (1977), and use a straight-wavefront approximation. This involves simply using a single azimuth (appropriate to the origin of the local coordinate system, say) for all the stations observing an earthquake. In this case, there still remains an error caused by the curvature of the wavefront, but this results from a truly seismological approximation and does not depend upon the latitude of the region being studied. Nevertheless, this error is often significant, and eliminating it using computed grid directions is a better approach.

Correcting for Grid-Direction Deviation

A more accurate approach is to take into account the curvature of the wavefront by computing the deviation between grid directions and true directions and correcting each observation for this deviation.

Consider a curve on the surface of a unit-radius sphere, specified parametrically by $\theta = \Theta(p)$ and $\phi = \Phi(p)$, where θ is the polar angle (colatitude), ϕ is longitude, and Θ and Φ are functions of p , which is distance along the curve. The azimuth, j , of the tangent to the curve in the direction of increasing p is related to these functions by the relations

$$\cos j = -\Theta'(p) \text{ and } \sin j = \Phi'(p) \sin \theta, \quad (1)$$

where the prime symbol indicates differentiation. Let this same curve be specified in the local rectangular (x, y) coordinate system by the parametric equations $x = X(q)$ and $y = Y(q)$, where q is arc length along the mapped curve. The mapping implicitly defines a functional relation, $q(p)$, which is strictly increasing, between the two parameters.

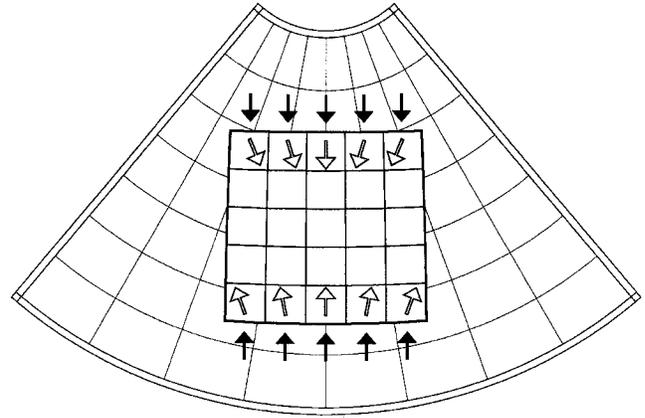


Figure 1. Relation between global (spherical) and local (Cartesian) coordinates, at a northern latitude. Black arrows show directions of approach of seismic waves from earthquakes at epicentral distances of 90° to the north and south. White arrows show artificial distortion of wavefronts that occurs if directions in the local rectangular grid are identified with directions in the spherical grid.

Then the grid azimuth, j_{grid} , of the tangent to the curve in the direction of increasing p is related to the functions $X(q)$ and $Y(q)$ by

$$\cos j_{\text{grid}} = X'(q) \text{ and } \sin j_{\text{grid}} = Y'(q). \quad (2)$$

(The x axis is directed upward, and the y axis is directed to the right.) Thus, to determine how directions on the sphere are distorted by the local coordinate system, we must evaluate the ratio of the derivatives $X'(q)$ and $Y'(q)$, given $\Theta'(p)$ and $\Phi'(p)$.

The local-grid coordinates are functions, $x(\theta, \phi)$ and $y(\theta, \phi)$, of the spherical coordinates θ and ϕ , which depend upon the particular local mapping chosen. If the two parametric representations given above are to represent the same curve, then we must have $x(\Theta(p), \Phi(p)) \equiv X(q(p))$ and $y(\Theta(p), \Phi(p)) \equiv Y(q(p))$. Differentiating with respect to p we get

$$X'(q)q'(p) = \frac{\partial x}{\partial \theta} \Theta'(p) + \frac{\partial x}{\partial \phi} \Phi'(p) \text{ and} \quad (3)$$

$$Y'(q)q'(p) = \frac{\partial y}{\partial \theta} \Theta'(p) + \frac{\partial y}{\partial \phi} \Phi'(p), \quad (4)$$

and using equations (1) and (2), these become

$$q'(p) \cos j_{\text{grid}} = -\frac{\partial x}{\partial \theta} \cos j + \frac{1}{\sin \theta} \frac{\partial x}{\partial \phi} \sin j \text{ and} \quad (5)$$

$$q'(p) \sin j_{\text{grid}} = -\frac{\partial y}{\partial \theta} \cos j + \frac{1}{\sin \theta} \frac{\partial y}{\partial \phi} \sin j. \quad (6)$$

These are the equations for transforming directions, and they are completely general. They can be used to compute the grid direction, j_{grid} , from the spherical direction, j , for any mapping, given the functions $x(\theta, \phi)$ and $y(\theta, \phi)$ that define the mapping. For this purpose, the (guaranteed positive) value of the common multiplier $q'(p)$ is not needed.

Azimuthal Mappings

The general equations for transforming directions, (5) and (6), can be specialized for our particular application, because all the mappings we consider are azimuthal. That is, they are all of the form

$$x(\theta, \phi) = f(\Delta)\cos \zeta, \quad y(\theta, \phi) = f(\Delta)\sin \zeta, \quad (7)$$

where $\Delta(\theta, \phi)$ and $\zeta(\theta, \phi)$ are the great-circle distance and bearing from the local origin (θ_0, ϕ_0) to the point (θ, ϕ) , and the projections differ only in the distance-mapping function $f(\Delta)$. Table 1 gives the function $f(\Delta)$ and its derivative for the five common azimuthal mappings mentioned earlier.

Using the special forms (7), the partial derivatives needed in the angle-transformation equations (5) and (6) can be expressed in terms of partial derivatives of the functions $\Delta(\theta, \phi)$ and $\zeta(\theta, \phi)$:

$$\frac{\partial x}{\partial \theta} = f'(\Delta)\cos \zeta \frac{\partial \Delta}{\partial \theta} - f(\Delta)\sin \zeta \frac{\partial \zeta}{\partial \theta}; \quad (8)$$

$$\frac{\partial y}{\partial \theta} = f'(\Delta)\sin \zeta \frac{\partial \Delta}{\partial \theta} + f(\Delta)\cos \zeta \frac{\partial \zeta}{\partial \theta};$$

$$\frac{\partial x}{\partial \phi} = f'(\Delta)\cos \zeta \frac{\partial \Delta}{\partial \phi} - f(\Delta)\sin \zeta \frac{\partial \zeta}{\partial \phi}; \quad (9)$$

$$\frac{\partial y}{\partial \phi} = f'(\Delta)\sin \zeta \frac{\partial \Delta}{\partial \phi} + f(\Delta)\cos \zeta \frac{\partial \zeta}{\partial \phi}.$$

By differentiating the spherical-trigonometry relations

$$\cos \Delta = \cos \theta_0 \cos \theta + \sin \theta_0 \sin \theta \cos(\phi - \phi_0), \quad (10)$$

$$\begin{aligned} \cos \zeta &= \frac{\cos \theta - \cos \theta_0 \cos \Delta}{\sin \theta_0 \sin \Delta} \\ &= \frac{\sin \theta_0 \cos \theta - \cos \theta_0 \sin \theta \cos(\phi - \phi_0)}{\sin \Delta}, \end{aligned} \quad (11)$$

and

$$\sin \zeta = \frac{\sin \theta \sin(\phi - \phi_0)}{\sin \Delta} \quad (12)$$

Table 1
Defining Functions for Azimuthal Mappings

Projection	$f(\Delta)$	$f'(\Delta)$
Equidistant	Δ	1
Orthographic	$\sin \Delta$	$\cos \Delta$
Gnomonic	$\tan \Delta$	$\sec^2 \Delta$
Stereographic	$2 \tan \frac{\Delta}{2} = \frac{2 \sin \Delta}{1 + \cos \Delta}$	$\sec^2 \frac{\Delta}{2} = \frac{2}{1 + \cos \Delta}$
Equal-Area	$\sqrt{2(1 - \cos \Delta)}$	$\frac{\sin \Delta}{\sqrt{2(1 - \cos \Delta)}}$

(Abramowitz and Stegun, 1964, Section 4.3.149) with respect to θ and ϕ , we obtain the partial derivatives needed in (8) and (9):

$$\frac{\partial \Delta}{\partial \theta} = \frac{\cos \theta_0 \sin \theta - \sin \theta_0 \cos \theta \cos(\phi - \phi_0)}{\sin \Delta}, \quad (13)$$

$$\frac{\partial \Delta}{\partial \phi} = \frac{\sin \theta_0 \sin \theta \sin(\phi - \phi_0)}{\sin \Delta}, \quad (14)$$

$$\begin{aligned} \sin \zeta \frac{\partial \zeta}{\partial \theta} &= \frac{\sin \theta \sin \Delta + (\cos \theta \cos \Delta - \cos \theta_0) \frac{\partial \Delta}{\partial \theta}}{\sin \theta_0 \sin^2 \Delta} \end{aligned} \quad (15)$$

$$\sin \zeta \frac{\partial \zeta}{\partial \phi} = \frac{(\cos \theta \cos \Delta - \cos \theta_0) \frac{\partial \Delta}{\partial \phi}}{\sin \theta_0 \sin^2 \Delta} \quad (16)$$

$$\begin{aligned} \cos \zeta \frac{\partial \zeta}{\partial \theta} &= \frac{\sin(\phi - \phi_0) \left(\cos \theta \sin \Delta - \sin \theta \cos \Delta \frac{\partial \Delta}{\partial \theta} \right)}{\sin^2 \Delta} \end{aligned} \quad (17)$$

and

$$\begin{aligned} \cos \zeta \frac{\partial \zeta}{\partial \phi} &= \frac{\sin \theta \left(\cos(\phi - \phi_0) \sin \Delta - \sin(\phi - \phi_0) \cos \Delta \frac{\partial \Delta}{\partial \phi} \right)}{\sin^2 \Delta}. \end{aligned} \quad (18)$$

(We use the apparently redundant last four of these equations, rather than just two of them, to avoid indeterminacy, which otherwise arises when either $\sin \zeta$ or $\cos \zeta$ equals

zero.) Equations (8) through (18) enable us to compute, for any point (θ, ϕ) , the partial derivatives appearing in the angle-transformation equations (5) and (6), so that we can use them to evaluate the sine and cosine of the grid direction j_{grid} from the corresponding functions of the spherical direction j .

Special Cases

For two of the azimuthal mappings we consider, it is possible to simplify the mathematical relations significantly.

Orthographic mapping. It can be shown using equations (8) through (18) that the derivatives needed in the angle-transformation equations (5) and (6) take the following simple forms:

$$\begin{aligned}\frac{\partial x}{\partial \theta} &= -\sin \theta_0 \sin \theta - \cos \theta_0 \cos \theta \cos(\phi - \phi_0); \\ \frac{\partial y}{\partial \theta} &= \cos \theta \sin(\phi - \phi_0); \end{aligned} \quad (19)$$

$$\begin{aligned}\frac{1}{\sin \theta} \frac{\partial x}{\partial \phi} &= \cos \theta_0 \sin(\phi - \phi_0); \\ \frac{1}{\sin \theta} \frac{\partial y}{\partial \phi} &= \cos(\phi - \phi_0). \end{aligned} \quad (20)$$

Stereographic mapping. Because this mapping is conformal, it preserves the angle at which any two curves cross each other. In other words, the transformation of directions described by equations (5) and (6) is simply a rotation through an angle that depends upon the location $(\theta$ and $\phi)$ but is independent of the direction j . This property follows from the Cauchy conditions

$$\frac{\partial x}{\partial \theta} = -\frac{1}{\sin \theta} \frac{\partial y}{\partial \phi} \text{ and } \frac{1}{\sin \theta} \frac{\partial x}{\partial \phi} = \frac{\partial y}{\partial \theta}, \quad (21)$$

which can be derived by substituting equations (13)–(18) and the definitions for the stereographic mapping from Table 1 into equations (8) and (9), and using equations (10)–(12). If we substitute the general angle-transformation formulas (5) and (6) into the trigonometric identities

$$\cos(j_{\text{grid}} - j) = \cos j_{\text{grid}} \cos j + \sin j_{\text{grid}} \sin j \quad (22)$$

and

$$\sin(j_{\text{grid}} - j) = \sin j_{\text{grid}} \cos j - \cos j_{\text{grid}} \sin j, \quad (23)$$

and use the Cauchy conditions, we get

$$q'(p) \cos(j_{\text{grid}} - j) = -\frac{\partial x}{\partial \theta} \quad (24)$$

and

$$q'(p) \sin(j_{\text{grid}} - j) = -\frac{\partial y}{\partial \theta}. \quad (25)$$

The partial derivatives on the right-hand sides of these equations are independent of the direction j , which shows that the Cauchy conditions imply conformality, so the stereographic mapping is conformal, as claimed. These partial derivatives can be evaluated using equations (8), the definitions of $f(\Delta)$ and $f'(\Delta)$ from Table 1, the trigonometric relations (10)–(12), and the derivatives (13)–(18). The resulting equations,

$$\begin{aligned}q'(p) \cos(j_{\text{grid}} - j) \\ = 2 \frac{\sin \theta \sin \theta_0 + \cos \theta_0 \cos \theta \cos \phi + \cos(\phi - \phi_0)}{(1 + \cos \Delta)^2} \end{aligned} \quad (26)$$

and

$$\begin{aligned}q'(p) \sin(j_{\text{grid}} - j) \\ = \frac{2(\cos \theta_0 + \cos \theta) \sin(\phi - \phi_0)}{(1 + \cos \Delta)^2}, \end{aligned} \quad (27)$$

can be used to evaluate the rotation angle $j_{\text{grid}} - j$ for this projection.

Ellipticity

The effect of the Earth's ellipticity of figure is probably negligible for regional tomographic applications, but it is possible to account for it approximately by using the geocentric colatitude, θ_c instead of the geographic colatitude, θ_g , in computations. These angles are related by

$$\cot \theta_c = (1 - \alpha)^2 \cot \theta_g, \quad (28)$$

where α is the flattening of the ellipsoid. For the WGS84 ellipsoid, $\alpha = 1/298.257$.

Concluding Remarks

A version of the computer program THRD incorporating a simplified version of the correction for the orthographic mapping was developed and used for the Iceland hotspot teleseismic tomography study of Foulger *et al.* (2000). This program is available via anonymous ftp from *swave.wr.usgs.gov* in the file *pub/outgoing/julian/thrd.tar.Z*.

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